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Proca and Electromagnetic Fields

P. Hillion[1] **and S. Quinnez**[2]

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In the framework of the proper orthochronous Lorentz group, the old connection is revived between the electromagnetic field characterized by a self-dual tensor and a traceless second-rank spinor obeying the Proca equation. The relationship between this spinor and the Hertz potential also considered as a self-dual tensor is emphasized. The extension of this formalism to meet the covariance under the full Lorentz group is also discussed.

1. INTRODUCTION

In the preface of their recent book, Penrose and Rindler, (1984) state;

> Spinor Calculus may be regarded as applying at a deeper level of structure of space-time than described by the standard world tensor calculus .... In fact any world tensor calculation can by an obvious prescription be translated into a 2-spinor form. The reverse is also, in a sense, true .... This effective equivalence may have led some "sceptics" to believe that spinors are "unnecessary." We hope that this book will help to convince the reader that there are many classes of spinorial results ... whose antecedents and interrelations would be totally obscured by tensor descriptions.

In fact we have been trying for some time to uphold similar ideas within the framework of electromagnetism. In this work we shall not be concerned with quantum mechanics and, unlike most authors [notable exceptions are Bateman (1914) and Laporte and Uhlenbeck (1931)], we use a complex three-vector (self-dual tensor) rather than a real, antisymmetric tensor to describe the electromagnetic field. We also use the Hertz potentials instead of the four-vector potential.

Then we discuss the equivalence between the 2-spinor field and the complex electromagnetic field, first in a pedestrian way and then in a more rigorous manner.

[1]Institut Henri Poincaré, 75231 Paris, France.
[2]Laboratoire de Physique-Mathematique, Faculté des Sciences Mirande, 21004 Dijon, France.

2. PEDESTRIAN APPROACH

We start with the Maxwell equations;

$$\mathscr{E}_{klm}\,\partial^l E^m = -\frac{\mu}{c}\,\partial_t H_k, \qquad \mathscr{E}_{klm}\,\partial^l H^m = \frac{\varepsilon}{c}\,\partial_t E_k, \qquad k, l, m = 1, 2, 3 \quad (1)$$

where $E_k = E_k(\bar{x}, t)$ and $H_k = H_k(\bar{x}, t)$ are the components of the electric and magnetic fields, $\mathscr{E}_{klm}$ is the permutation tensor, ε and μ are the permittivity and the permeability, respectively, c is the velocity of light, ∂_j and ∂_t are the derivatives with respect to x_j and t, respectively, and $\bar{x}$ is an arbitrary point in R^3.

If we introduce the complex vector

$$\Lambda_j = -(\sqrt{\varepsilon}\,E_j - i\sqrt{\mu}\,H_j), \qquad i = \sqrt{-1}, \qquad j = 1, 2, 3 \quad (2)$$

equations (1) become (n is the refractive index)

$$i\mathscr{E}_{klm}\,\partial^k \Lambda^m = \frac{n}{c}\,\partial_t \Lambda_k, \qquad k = 1, 2, 3 \quad (1')$$

From (1), we get the following set of equations:

$$\pm i\varepsilon\,\partial_t(E_x \pm iE_y) + \partial_z(H_x \pm iH_y) = (\partial_x \pm i\partial_y)H_z$$

$$\partial_z(E_x \pm iE_y) \mp i\frac{\mu}{c}\,\partial_t(H_x \pm iH_y) = (\partial_x \pm i\partial_y)E_z$$

which becomes in terms of Λ_j

$$\begin{aligned} (\partial_x + i\partial_y)\Lambda_z &= \left(\partial_z + \frac{n}{c}\,\partial_t\right)(\Lambda_x + i\Lambda_y) \\ (\partial_x - i\partial_y)\Lambda_z &= \left(\partial_z - \frac{n}{c}\,\partial_t\right)(\Lambda_x - i\Lambda_y) \end{aligned} \quad (3)$$

Let us now consider a set of two spinors $\Psi^a(\bar{x}, t)$, $a = 1, 2$, with complex components $\psi^a_\alpha(\bar{x}, t)$, $\alpha = 1.2$, satisfying the Pauli equation;

$$\left(\sigma^j\,\partial_j - \frac{n}{c}\,\partial_t\right)\Psi^a(\bar{x}, t) = 0, \qquad a = 1, 2 \quad (4)$$

σ_j are the Pauli matrices, and we use the summation convention $\sigma^j\partial_j = \sigma_1\partial_x + \sigma_2\partial_y + \sigma_3\partial_z$ with the following representation of the Pauli matrices:

$$\sigma_1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \qquad \sigma_2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \qquad \sigma_3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}$$

Equation (4) takes the form

$$
\begin{aligned}
&(\partial_x + i\partial_y)\psi_1^a - \left(\partial_z + \frac{n}{c}\partial_t\right)\psi_2^a = 0 \\
&(\partial_x - i\partial_y)\psi_2^a + \left(\partial_z - \frac{n}{c}\partial_t\right)\psi_1^a = 0
\end{aligned} \tag{5}
$$

The comparison of (3) and (5) leads to the identifications

$$
\psi_1^1 = \Lambda_z, \qquad \psi_2^1 = \Lambda_x + i\Lambda_y \qquad \psi_2^2 = -\Lambda_z, \qquad \psi_1^2 = \Lambda_x - i\Lambda_y \tag{6a}
$$

or

$$
\Lambda_x = \tfrac{1}{2}(\psi_2^1 + \psi_1^2), \qquad \Lambda_y = \tfrac{1}{2i}(\psi_2^1 - \psi_1^2), \qquad \Lambda_z = \tfrac{1}{2}(\psi_1^1 - \psi_2^2) \tag{6b}
$$

with the constraint on the spinor field

$$
\psi_1^1 + \psi_2^2 = 0 \tag{7}
$$

If one denotes by $\sigma_{j\alpha}^a$ the components of the matrix σ_j, the relations (6a) and (6b) may be written in a more compact form:

$$
\psi_\alpha^a = \sigma_{j,\alpha}^a \Lambda^j \tag{8a}
$$

$$
\Lambda_j = \tfrac{1}{2}\sigma_{j,a}^\alpha \psi_\alpha^a \tag{8b}
$$

with summation on j in (8a) and on the indices a, α in (8b). Using (6) and (7), it is trivial to prove the equivalence of equations (1′) and (4).

The Poynting vector S_j and the energy density W of the electromagnetic field are defined by the relations

$$
\begin{aligned}
S_j &= \frac{ic}{4\pi n}\,\mathscr{E}_{jkl}\Lambda^{*kl} = \frac{c}{16\pi}\sum_{a=1}^{2}(\Psi^{a+}\sigma_j\Psi^a) \\
W &= -\Lambda_j^*\Lambda^j = \frac{1}{16}\sum_{a=1}^{2}\Psi^{a+}\Psi^a
\end{aligned} \tag{9}
$$

and they satisfy the conservation equation $\partial^j S_j - n\,\partial_t W = 0$. The asterisk denotes the complex conjugation and Ψ^{a+} is the Hermitian conjugated spinor.

Let us now introduce a scalar φ such that

$$
\Lambda_z = \psi_1^1 = \left(\partial_z^2 - \frac{n^2}{c^2}\partial_t^2\right)\varphi \tag{10}
$$

Then, using (5) and (6), we get

$$
\begin{aligned}
\Lambda_x &= \frac{1}{2}\left[(\partial_x + i\partial_y)\left(\partial_z - \frac{n}{c}\right)\varphi + (\partial_x - i\partial_y)\left(\partial_z + \frac{n}{c}\partial_t\right)\varphi\right] \\
\Lambda_y &= \frac{1}{2i}\left[(\partial_x + i\partial_y)\left(\partial_z - \frac{n}{c}\partial_t\right)\varphi - (\partial_x - i\partial_y)\left(\partial_z + \frac{n}{c}\partial_t\right)\varphi\right]
\end{aligned} \tag{10'}
$$

Define the vector Ξ by $\Xi = \varphi K$, where K is a unit vector along the z axis; then from (10) and (10′) we get at once

$$\Xi = \varphi K = \Pi + iM \tag{11}$$

where Π and M are, respectively, the electric and magnetic Hertz vector (Jones, 1964). Since no special choice of z axis has been made, it follows that any electromagnetic field in a homogeneous isotropic medium in free space, free from charges and currents, can be expressed either in terms of the spinors Ψ^a or in terms of the complex Hertz vector Ξ.

We have been using for some time the spinor formalism to discuss electromagnetic beam propagation particularly in the domain of guided waves (Hillion and Quinnez, 1985a, 1986a) and optical fibers (Hillion and Quinnez, 1986b). Let us only mention, for instance, that assuming propagation in the Oz direction, the TM and TE modes correspond, respectively, to Im $\varphi = 0$ and Re $\varphi = 0$ in (10).

An interesting case is when one has

$$\begin{aligned} \psi_1^1 &= f_1 g_1, \qquad \psi_1^2 = f_1 g_2 \\ \psi_2^1 &= f_2 g_1 \qquad \psi_2^2 = f_2 g_2 \end{aligned} \tag{12}$$

where (f_1, f_2) and (g_1, g_2) are the components of spinors satisfying the Pauli equation.

The condition (7) becomes

$$f_1 g_1 + f_2 g_2 = 0 \tag{13}$$

By substituting (12) into (6) and using (13), it is easy to show that Λ_j is a null vector $\Lambda_j \Lambda^j = 0$, which implies, according to (2),

$$\varepsilon |E|^2 = \mu |H|^2, \qquad E \cdot H = 0 \tag{14}$$

This situation corresponds to the TEM modes for guided waves or to the Bateman waves (Hogan, 1984) in a more general context. In fact we started our work with this particular case (Hillion and Quinnez, 1985b–d).

3. RIGOROUS APPROACH

3.1. Proca Field

The pedestrian approach is unsatisfactory, since we did not justify the introduction of the two spinors Ψ^a. But everything becomes clear in a relativistic framework; consider a traceless second-rank spinor $\Psi^r{}_s$ $(r, s = 1, 2)$ and the Proca equation (Corson, 1954; Umezawa, 1956):

$$\partial^{rs} \psi_r^t = 0 \tag{15}$$

with

$$\partial^{r\dot{s}} = \sigma^{r\dot{s}}_{\mu}\partial^{\mu}, \qquad \sigma_{\mu} = \{\sigma_j, \sigma_0\}, \qquad \partial_{\mu} = \left\{\partial_j, \partial_0 = \frac{n}{c}\partial_t\right\}, \qquad j = 1, 2, 3 \tag{15'}$$

As is well known (Corson, 1954; Umezawa, 1956), the Proca spinor field that corresponds to particles with spin one is a second-rank spinor with zero trace transforming according to the $\mathfrak{D}(1, 0)$ representation of the unimodular group $SL(2, C)$.

Moreover, it defines a self-dual tensor characterizing the electromagnetic field. As a consequence, the complex quantities Λ_j must be considered as the components $(0, j)$ and (k, l) of a self-dual tensor $\tilde{F}_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3$).

The connection between $\tilde{F}_{\mu\nu}$ and ψ^t_s is

$$\tilde{F}_{\mu\nu} = \sigma^{r}_{\mu s}\sigma^{t}_{\nu r}\psi^{s}_{t}, \qquad \mu, \nu = 0, 1, 2, 3 \tag{16}$$

and the Maxwell equations (1′) take the form

$$\mathscr{E}_{\mu\nu\rho\sigma}\,\partial^{\nu}\tilde{F}^{\rho\sigma} = 0, \qquad \mu = 0, 1, 2, 3 \tag{17}$$

$\mathscr{E}_{\mu\nu\rho\sigma}$ is the Ricci tensor equal to 1 (resp. −1) for an even (resp. odd) parity of the permutation μ, ν, ρ, σ of the indices 0, 1, 2, 3 and equal to zero if two or more indices are zero.

It is easy to prove the covariance of both equations (15) and (17) under the orthochronous proper Lorentz group $\mathscr{L}^{\uparrow}_{+}$ as well as the equivalence of both formalisms.

The complex Hertz vector Ξ_j must also be considered as a self-dual tensor $\tilde{\Xi}_{\mu\nu}$ which defines a traceless second-rank spinor $\mathscr{E}^r_s$ through relations similar to (16). Then on the basis of the covariant requirement it is easy to obtain ψ^r_s in terms of $\mathscr{E}^r_s$:

$$\psi^r_s = \partial_{r\dot{\lambda}}\partial^{\dot{\lambda}t}\mathscr{E}^s_t + \partial^{s\dot{\lambda}}\partial_{\dot{\lambda}t}\mathscr{E}^t_r \tag{18}$$

with the derivative operators defined in (15′). It is trivial to show that ψ^s_r as defined by (18) is traceless.

Remark 1. We conform to tradition by calling (15) a Proca equation, but in fact Laporte and Uhlenbeck (1931) used it some years before with symmetric spinors $\psi_{rs} = \varepsilon_{rt}\psi^t_s$, where ε_{rt} is the metric tensor of the spinor geometry (Corson, 1954; Umezawa, 1956). They also noticed the relations (16) and (18). The pedestrian approach of the previous section suggests, that the traceless spinors are more useful for practical applications [not considered in Laporte and Uhlenbeck (1931)].

Remark 2. When the second-rank spinor ψ^r_s degenerates into a product of two first-rank spinors we obtain the situation described previously in Corson (1954) and Eriksen (1960).

3.2. Proca–Pauli Field

At this point we may note that the spinor formalism of the last section is not exactly the same as the formalism supplied by the pedestrian approach. Moreover, it has two drawbacks; (1) it is difficult to take the traceless condition into account; and (2) the only Lorentz covariant that can be formed with ψ^r_s is the tensor $\psi^{\dot{u}}_{\dot{r}}\sigma_{\mu\dot{u}t}\sigma^{\dot{r}s}_{\nu}\psi^t_s$, which leads to some difficulties in obtaining a scalar Lagrangian density. ($\psi^{\dot{u}}_{\dot{r}}$ is the complex conjugate spinor). To cope with these facts, we introduce the matrix Ω and the unit spinors Φ_0 and $\hat{\Phi}_0$:

$$\Omega = \begin{vmatrix} \psi^1_1 & \psi^2_1 \\ \psi^1_2 & \psi^2_2 \end{vmatrix}, \qquad \Phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{\Phi}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{19}$$

and we define the Proca-Pauli field Ψ^a, $a = 1, 2$, by the relations $\Psi^1 = \Omega\Phi_0$ and $\Psi^2 = \Omega\hat{\Phi}_0$, so that equation (15) becomes the Proca-Pauli equation $\sigma^\mu\partial_\mu\Psi^a = 0$, $a = 1, 2$, which is nothing else than equation (4) written in a manifestly covariant form, it is easy to prove that Ψ^1 and Ψ^2 are first-rank spinors, so that we regain the pedestrian approach.

Then the Lagrangian density

$$\mathscr{L} = \frac{ic}{2}\sum_{a=1}^{2} (\Psi^{a+}\sigma^\mu\partial_\mu\Psi^a - \partial_\mu\Psi^{a+}\sigma^\mu\Psi^a) \tag{20}$$

is a real, scalar invariant under the proper orthochronous Lorentz group $L^\uparrow_+$. We may remark that $\mathscr{L}$ has the dimension of a power density rather than of an energy density as usual. The variation of $\mathscr{L}$ with respect to Ψ^{a+} supplies the Proca–Pauli equation.

Let j_μ be the energy flow vector: from (20) we get

$$j_\mu = \sum_{a=1}^{2} \Psi^{a+}\sigma_\mu\Psi^a, \qquad \mu = 0, 1, 2, 3 \tag{21}$$

which satisfies the conservation equation $\partial^\mu j_\mu = 0$. In agreement with (9), the Poynting vector and the electromagnetic energy density are, respectively, the components j_β and j_0 of j_μ.

The energy-momentum tensor $T_{\mu\nu}$ deduced from (21) is

$$T_{\mu\nu} = \frac{ic}{2}\sum_{a=1}^{2} (\Psi^{a+}\sigma_\mu\partial_\nu\Psi^a - \partial_\nu\Psi^{a+}\sigma_\mu\Psi^a) \tag{22}$$

In particular, one has

$$T_{0\nu}=\frac{ic}{2}\sum_{a=1}^{2}\left[\Psi^{a+}\partial_\nu\Psi^a-(\partial_\nu\Psi^{a+})\Psi^a\right]$$

and a simple calculation gives

$$T_{0\nu}=-2nc(H_k\,\partial_\nu E^k-E_k\,\partial_\nu H^k) \tag{23}$$

$T_{0\nu}$ satisfies the conservation equation $\partial^\nu T_{0\nu}=0$, and for $\nu=0$ we get

$$\begin{aligned} T_{00}&=-2nc(H_k\,\partial_0 E_k-E_k\,\partial_0 H^k)\\ &=-c(\mu H\cdot \text{curl } H+\varepsilon E\cdot \text{curl } E) \end{aligned} \tag{24}$$

To discuss the physical meaning of this energy-momentum tensor, we need some results from the vector field theory. Let Γ be a twice continuously differentiable vector field, the quantity $\lambda=\Gamma\cdot\text{curl}\,\Gamma$, where the dot means the usual scalar product, is called the abnormality of the field (Eriksen, 1960), λ may be regarded as a measure of the departure of Γ from the property of having a normal congruence of surfaces. So, according to (24), the energy-momentum tensor has to do with the vorticity of the electromagnetic field.

To make precise this idea, let us consider an electromagnetic screwfield (Eriksen, 1960) of constant abnormality. Such a field is characterized by the relations

$$H\wedge\text{curl } H=0,\qquad E\wedge\text{curl } E=0 \tag{25}$$

where the symbol $\wedge$ denotes the outer product. This implies

$$\text{curl } H=\lambda H,\qquad \text{curl } E=\lambda E \tag{25$'$}$$

and taking the curl of (25′) gives

$$\Delta H+\lambda^2 H=0,\qquad \Delta E+\lambda^2 E=0 \tag{26}$$

where Δ is the Laplacian operator. Now from (1), (25), and (25′), we get

$$H\wedge\frac{dE}{dt}=0=E\wedge\frac{dH}{dt} \tag{27a}$$

$$\lambda E=-\frac{\mu}{c}\frac{dH}{dt},\qquad \lambda H=\frac{\varepsilon}{c}\frac{dE}{dt} \tag{27b}$$

and these relations imply

$$\frac{d^2H}{dt^2}+\frac{\lambda^2c^2}{n^2}H=0,\qquad \frac{d^2E}{dt^2}+\frac{\lambda^2c^2}{n^2}E=0 \tag{28}$$

Eliminating λ^2 between (26) and (28) leads to the wave equations. Now a solution of (27) and (28) takes the form

$$E = \frac{A^1}{\sqrt{\varepsilon}} \cos \frac{\lambda n}{c} t + \frac{A^2}{\sqrt{\varepsilon}} \sin \frac{\lambda n}{c} t$$
$$H = \frac{A^1}{\sqrt{\mu}} \cos \frac{\lambda n}{c} t - \frac{A^2}{\sqrt{\mu}} \sin \frac{\lambda n}{c} t \tag{29}$$

where A^1 and A^2 are two vectors compelled to satisfy the following relations:

$$\text{curl } A^1 = \lambda A^1, \qquad \text{curl } A^2 = \lambda A^2, \qquad \text{div } A^1 = 0 = \text{div } A^2 \tag{30}$$

in order for (25′) to be fulfilled. Substituting (29) into (24) shows that T_{00} is the power density of the electromagnetic screw-field, while from (23) and (29) we get

$$T_{0j} = 2c(A^1 \partial_j A^2 - A^2 \partial_j A^1), \qquad j = 1, 2, 3 \tag{31}$$

But still using (29) and taking (30) into account, we find the Poynting vector as

$$S = \frac{c}{2n} (A^1 \wedge A^2) = \frac{c}{2n\lambda} (\text{curl } A^1 \wedge A^2 + A^1 \wedge \text{curl } A^2)$$

and simple calculation gives

$$S_j = \frac{c}{2n\lambda} (A^1 \partial_j A^2 - A^2 \partial_j A^1) + + \frac{1}{2\lambda} (\text{curl } S)_j$$
$$= \frac{T_{0j}}{4n\lambda} + \frac{1}{2\lambda} (\text{curl } S)_j \tag{32}$$

So T_{0j} is defined in terms of the Poynting vector and of its vorticity. It is attractive to speculate about the existence of electromagnetic screw-fields.

4. DISCUSSION

The previous formalism works very well as long as one only needs invariance under the orthochronous Lorentz group $L_+^\uparrow$. This situation occurs in classical mechanics, where we may agree to write equations in a right-handed reference frame with positive time going on into the future. In this case either the 2-spinor or the self-dual tensor formalism may be simpler than the usual formalism with a real antisymmetric tensor or a set of two three-vectors (Bateman, 1914; Hillion and Quinnez, 1985a, b, 1986a, b; Hogan, 1984). Also, in many cases, the Hertz potential is more useful than the real four-vector potential (Jones, 1964; Weeks, 1968; Kerber, 1969; Kelso, 1964).

The situation is a bit different when one needs invariance under the full Lorentz group, which requires in addition to $\psi^r{}_s$ and $\tilde{F}_{\mu\nu}$ the complex conjugate fields $\psi^{\dot{r}}{}_{\dot{s}}$ and $\tilde{F}^*_{\mu\nu}$. There is no problem in the tensorial case, since the electromagnetic field tensor $F_{\mu\nu}$ satisfies $F_{\mu\nu}=\frac{1}{2}(\tilde{F}_{\mu\nu}+\tilde{F}^*_{\mu\nu})$. In the spinor case, many authors [for instance, Moses (1959), Good (1957), and Campolattaro (1980), and references given therein], starting with the results of Laporte and Uhlenbeck, have been trying with a mixed success to develop a 4-spinor formalism of electromagnetism. We show that the Proca–Pauli fields may lead to an elegant solution of this problem.

To be clear, we repeat some of the previous results. The two sets of Proca's equations are

$$\partial^{r\dot{s}}\psi^t_r=0, \qquad \partial_{r\dot{s}}\psi^{\dot{s}}_t=0 \tag{33}$$

We introduce the traceless matrix Ω and its Hermitian conjugate Ω^+ together with the unit spinors Φ_0 and $\hat{\Phi}_0$:

$$\Omega=\begin{vmatrix}\psi^1_1 & \psi^2_1\\ \psi^1_2 & \psi^2_2\end{vmatrix}, \qquad \Omega^+=\begin{vmatrix}\psi^{\dot{1}}_{\dot{1}} & \psi^{\dot{1}}_{\dot{2}}\\ \psi^{\dot{2}}_{\dot{1}} & \psi^{\dot{2}}_{\dot{2}}\end{vmatrix}, \qquad \Phi_0=\begin{bmatrix}1\\0\end{bmatrix}, \qquad \hat{\Phi}_0=\begin{bmatrix}0\\1\end{bmatrix} \tag{34}$$

We define the first-rank spinors Ψ^a, $\Psi_{\dot{a}}$, a, $\dot{a}=1,2$, transforming, respectively, according to the representations $\mathfrak{D}(\frac{1}{2},0)$ and $\mathfrak{D}(0,\frac{1}{2})$ of the $SL(2, \mathrm{C})$ group:

$$\Psi^1=\Omega\Phi_0, \qquad \Psi^2=\Omega\hat{\Phi}_0, \qquad \Psi_{\dot{1}}=\Omega^+\Phi_0, \qquad \Psi_{\dot{2}}=\Omega^+\hat{\Phi}_0 \tag{35}$$

From (33) and using the well-known relations

$$\partial^{1\dot{1}}=\partial_{2\dot{2}}, \qquad \partial^{2\dot{2}}=\partial_{1\dot{1}}, \qquad \partial^{1\dot{2}}=-\partial_{2\dot{1}}, \qquad \partial^{2\dot{1}}=-\partial_{1\dot{2}}$$

one proves easily that the spinors Ψ^a and Ψ_a satisfy the equations

$$\begin{aligned}(\sigma^j\partial_j-\sigma_0\partial_0)\Psi^a&=0, \qquad a=1,2\\ (\sigma^j\partial_j+\sigma_0\partial_0)\Psi_{\dot{a}}&=0, \qquad \dot{a}=1,2\end{aligned} \tag{36}$$

Let Φ^a, $a=1,2$, be the two 4-spinors

$$\Phi^1=\begin{bmatrix}\Psi^1\\ \Psi_{\dot{1}}\end{bmatrix}=\begin{bmatrix}\Psi^1_1\\ \Psi^1_2\\ \Psi^{\dot{1}}_{\dot{1}}\\ \Psi^{\dot{2}}_{\dot{1}}\end{bmatrix}, \qquad \Phi^2=\begin{bmatrix}\Psi^2\\ \Psi_{\dot{2}}\end{bmatrix}=\begin{bmatrix}\Psi^2_1\\ \Psi^2_2\\ \Psi^{\dot{1}}_{\dot{2}}\\ \Psi^{\dot{2}}_{\dot{2}}\end{bmatrix} \tag{37}$$

They satisfy the Dirac equation $\gamma^\mu\partial_\mu\Phi^a=0$, $a=1,2$, with, according to (36), the following representation of the matrices γ:

$$\gamma_j=\begin{vmatrix}0 & \sigma_j\\ \sigma_j & 0\end{vmatrix}, \qquad \gamma_0=\begin{vmatrix}0 & -\alpha_0\\ \sigma_0 & 0\end{vmatrix} \tag{38}$$

We shall discuss elsewhere the applications of this formalism.

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